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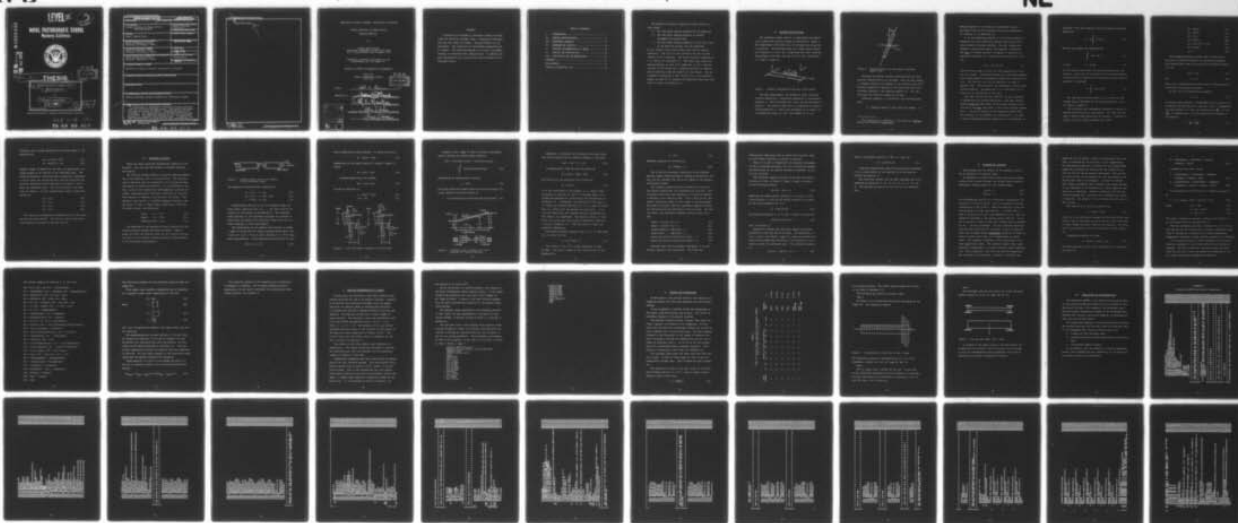
NAVAL POSTGRADUATE SCHOOL MONTEREY CALIF  
LATERAL BUCKLING OF BEAMS WITHOUT WARPING RIGIDITY.(U)  
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9 Master's THESIS,

6 LATERAL BUCKLING OF BEAMS WITHOUT  
WARPING RIGIDITY,

by

10 Robert Edward Brown

11 Jun 78

12 58 p.

Thesis Advisor: J. E. Brock

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Lateral Buckling of Beams Without Warping Rigidity		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis; June 1978
7. AUTHOR(s) Robert Edward Brown		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		8. CONTRACT OR GRANT NUMBER(s)
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Naval Postgraduate School Monterey, California 93940		12. REPORT DATE June 1978
		13. NUMBER OF PAGES 58
		15. SECURITY CLASS. (of this report) Unclassified
		16a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  Lateral buckling; lateral instability, buckling of beams.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  A procedure is described to determine lateral buckling loads for initially straight beams. Loading and beam geometry may vary along the length. Warping rigidity is not considered. End conditions of considerable generality may be treated. The algorithm depends on solving a convergent sequence of sixteenth order eigenproblems. A computer program implementing this procedure has been developed and is		



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Lateral Buckling of Beams Without  
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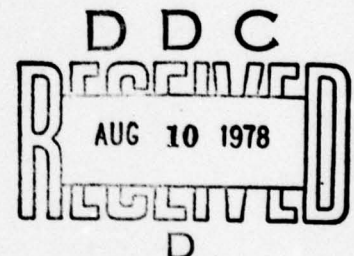
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Submitted in partial fulfillment of the  
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MASTER OF SCIENCE IN MECHANICAL ENGINEERING

from the  
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ACCESSION NO.	
RTIS	White Section <input checked="" type="checkbox"/>
DDC	Full Section <input type="checkbox"/>
UNCLASSIFIED	<input type="checkbox"/>
CLASSIFICATION	<input type="checkbox"/>
BY	
DISTRIBUTION/AVAILABILITY CODES	
DEL	AVAIL. CODE/SPECIAL
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# ABSTRACT

A procedure is described to determine lateral buckling loads for initially straight beams. Loading and beam geometry may vary along the length. Warping rigidity is not considered. End conditions of considerable generality may be treated. The algorithm depends on solving a convergent sequence of sixteenth order eigenproblems. A computer program implementing this procedure has been developed and is described herein.

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The problem of analysis logically divides itself into three cases.

- 1.) The case where warping rigidity may be neglected.
- 2.) The case where warping rigidity is nonzero throughout the entire beam.
- 3.) The case where warping rigidity is zero over a part of the beam and nonzero over the remainder.

As will become clear, these three cases must be treated separately. The third is most difficult and has not been treated in the literature. The second has been addressed by J. E. Brock; cf. reference (1). The first case, where the warping rigidity is zero or is neglected, is not a special instance of the second and a separate analysis is required. Such an analysis forms the subject of this thesis. The development is parallel to that of Brock (1) to an extensive degree and many of the figures and equations herein are identical to those in reference (1).

## II. PROBLEM SPECIFICATIONS

The coordinate system used is a right-handed orthogonal  $x, y, z$  system with positive senses as indicated in figure 1. The longitudinal centroidal axis of the beam lies along the  $+z$  direction. The principal axes of a normal cross section are originally in the  $x$  and  $y$  directions but after torsional displacement (rotation) they are in the  $\xi$  and  $\eta$  directions, cf. figure 2 (page 10).

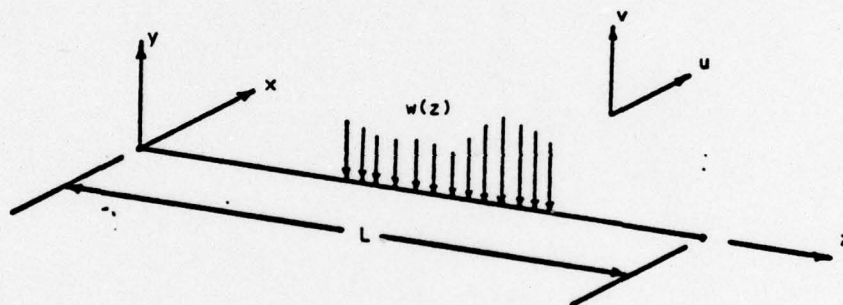


Figure 1. General configuration and coordinate system.

The beam displacements are defined by three variables: vertical deflection  $v$ , horizontal deflection  $u$ , and angular rotation  $\phi$ . Positive senses for  $v$  and  $u$  are as indicated in figure 1. The positive sense for  $\phi$  is indicated in figure 2. The deformations  $u$ ,  $v$  and  $\phi$  are considered to be "small" in a mathematical sense, so that, for example  $\sin \phi \approx \phi$ .

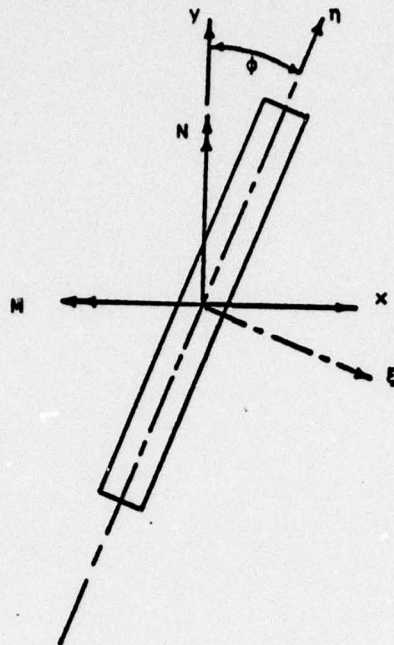


Figure 2. Positive sense of  $\phi$  and source of bending moment  $N-\phi M$ .

Defining the problem requires specifications for five physical characteristics of the beam. They are beam length  $L$ , flexural rigidity for bending in the vertical plane  $EI_1$ ,<sup>\*</sup> flexural rigidity for bending in the horizontal plane  $EI_2$ , torsional rigidity  $C$ , and warping rigidity  $C_1$ . The last four may vary along the length of the beam.

Torsional rigidity  $C$  is defined by the following equation.

$$C = (\text{TORQUE})/(\text{ANGLE OF TWIST PER UNIT LENGTH}) \quad (1)$$

---

<sup>\*</sup>For simplicity in notation, we use numerical suffixes rather than numerical subscripts.



Warping rigidity  $C_1$  is related to nonuniform torsion. A discussion here of this property would be too digressive; cf. section 5.3 of reference (2).

It is the underlying purpose of this entire analysis to determine the total externally applied load  $P$  which will cause incipient lateral buckling. The only loading considered is distributed load in the negative  $y$  direction. The type of loading (uniform, triangular or whatever) is specified beforehand by a dimensionless function  $f(z)$  such that

$$w(z) = (R/L^3) f(z) \quad (2)$$

where the load shape function  $w(z)$  has dimensions of force per unit length. Concentrated loads and concentrated moments can be approximated by appropriate large local variations in  $f(z)$ . The function  $f(z)$  will be denoted by  $x_5$  when it appears later. The constant  $R$  has dimensions of force times distance squared. The magnitude of  $R$  is arbitrary and is chosen at the convenience of the user.

The load function has been defined. We are interested in determining the multiplying factor  $Q$  such that lateral buckling does not take place if the actual loading is less than  $Qw(z)$  and does take place if the actual loading exceeds  $Qw(z)$ . The entire purpose of the program developed from this analysis is to determine this multiplier  $Q$ . In order to obtain nondimensionalized equations, it is convenient to

introduce  $P$  the total applied load for incipient buckling, defined by

$$P = Q \int_0^L w(z) dz \quad (3)$$

We will also employ the normalization

$$\int_0^L f(z) dz = L \quad (4)$$

so that  $Q = PL^2/R \quad (5)$

For the special case where the load contributions in the negative  $y$  direction are canceled by the load contributions in the positive  $y$  direction the following alternate form of equation 3 is applied.

$$P = \int_0^L Q |w(z)| dz \quad (6)$$

The load may be applied above, on, or below the centroidal axis as specified by the load eccentricity,  $e(z)$ , cf. figure 6 (page 20).

For notational and programming convenience a series of dimensionless functions  $x$  are defined. The first six are used to specify beam properties and loading. A seventh is simply a function having constant unit value.

$$x_1 = EI_1/R \quad (7)$$

$$x_2 = EI_2/R \quad (8)$$

$$x_3 = C/R \quad (9)$$

$$x_4 = RL^2/C_1 \quad (10)$$

$$x_5 = w(z) L^3/R = f(z) \quad (11)$$

$$x_6 = e(z)/L \quad (12)$$

$$x_7 = 1 \quad (13)$$

The nondimensionalizing scheme used to define the  $x$  functions is applied throughout the problem development. The dimensionless forms for vertical and horizontal deflection are

$$x_{10} = v/QL \quad (14)$$

and  $\beta = u/L \quad (15)$

where  $\beta$  is a dimensionless function which ultimately will be constructed from several functions.

We will use the dimensionless parameter

$$\zeta = z/L \quad (16)$$

to specify axial position. Accordingly we will think of the  $x$  functions as functions of  $\zeta$ , thus  $x_5 = x_5(\zeta)$ . Differentiation with respect to  $z$  and  $\zeta$  are indicated as  $(\frac{dY}{dz} = Y')$  and  $(\frac{dY}{d\zeta} = \dot{Y})$  respectively. The two operations are related by the equation

$$\frac{d}{dz} = \frac{1}{L} \frac{d}{d\zeta} \quad (17)$$



The developments which follow require considerable integration. Two notational devices are used to facilitate both the readability and the transcription of equations. Integration is signified by p, thus

$$px3 = \int_0^{\zeta} x3(\zeta)d\zeta .$$

Evaluation of any function at the right end of the beam, i.e., for  $\zeta = 1$ , is indicated by the prefix \*. Several examples illustrating this notation are

$$*px1 = \int_0^1 x1(\zeta)d\zeta \quad (18)$$

$$*x5x9 = x5(1) \cdot x9(1) \quad (19)$$

$$*x2 px8 = x2(1) \int_0^1 x8(\zeta)d\zeta \quad (20)$$

$$*px5x6px11 = \int_0^1 x5(\zeta)x6(\zeta) \left[ \int_0^{\zeta} x11(\theta)d\theta \right] d\zeta \quad (21)$$

(In the last equation  $\theta$  is a dummy variable.)

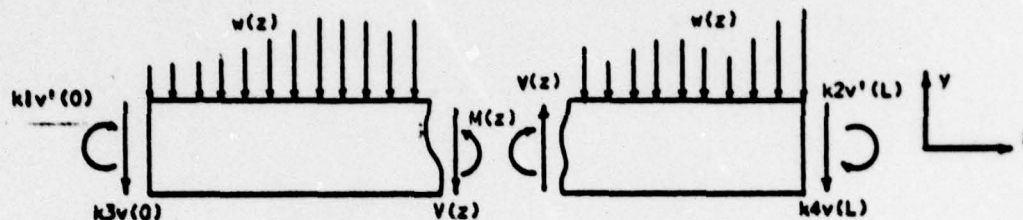


Figure 3. Diagram of forces and moments in the vertical plane.

Our analysis is restricted to beams which are unconstrained except at one or both ends. There are six possible constraints at each end each of which is modeled as a spring. The spring constant  $k$  describing such a constraint can actually vary from zero (perfectly free, no constraint at all) to infinity (perfect or ideal constraint) inclusive. Therefore to avoid mathematical difficulties the analysis employs dimensionless spring parameters, designated by the symbol  $a$  (with an appropriate identifying suffix). For a torsion spring, such as that corresponding to the moment  $k_1 v'(0)$  shown in figure 3 (page 15), the relations between  $k_1$  and  $a_1$  are

$$a_1 = k_1 / (k_1 + R/L) \quad (22)$$

$$k_1 = R a_1 / L (1 - a_1) \quad (23)$$

Note that  $a_1 = 0$  corresponds to  $k_1 = 0$  (perfect freedom) and  $a_1 = 1$  corresponds to  $k_1 = \infty$  (perfect constraint).

Similarly, for a linear spring such as spring number 3, the relations are

$$a_3 = k_3 / (k_3 + R/L^3) \quad (24)$$

$$k_3 = Ra_3/L^3(1 - a_3) \quad (25)$$

Figure 5 (page 19) shows that the point of application of a linear spring is not limited to the centroidal axis. The vertical deflection constraints may be applied at distances  $c_1$  and  $c_2$  above the centroidal axis, and the horizontal deflection constraints may be applied at distances  $c_3$  and  $c_4$  above the centroidal axis. The positive sense is as indicated in figure 5, and the dimensionless forms of these distances are

$$G_1 = c_1/L \quad (26)$$

$$G_2 = c_2/L \quad (27)$$

$$G_3 = c_3/L \quad (28)$$

$$G_4 = c_4/L \quad (29)$$

The physical and geometrical parameters used in the problem have now been defined. They will be used in the structural analysis developed in the next section.



### III. STRUCTURAL ANALYSIS

There are three structural deformations examined in the analysis. They are vertical bending, horizontal bending, and torsion.

The vertical bending problem is disjoint from the remainder of the analysis, and it is solved first. The notation used in defining vertical bending will not be shown because the problem is simple and familiar. It is sufficient to say that a set of four simultaneous nonhomogeneous equations are formed, representing the constraint conditions at each end which pertain to bending in the vertical (yz) plane. This system is then solved by a standard algebraic analysis, and the result is used to create three dimensionless functions for shear, moment, and deflection.

$$\text{Shear} \quad x_8 = V/P \quad (30)$$

$$\text{Moment} \quad x_9 = M/PL \quad (31)$$

$$\text{Deflection} \quad x_{10} = v/QL \quad (32)$$

The remainder of the analysis involves studying the combined horizontal bending and torsion problem. Figure 4 (page 18) shows the positive sense for the constant horizontal shear  $H$ , the linearly varying horizontal bending moment  $N$ , and horizontal displacement  $u$ .

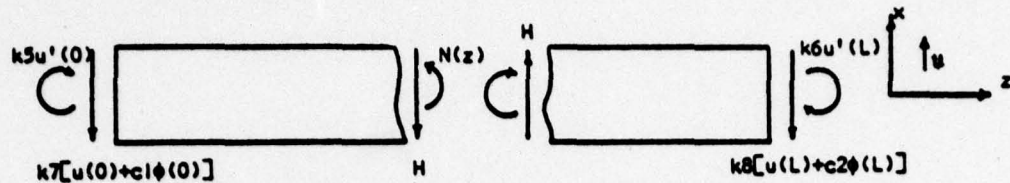


Figure 4. Diagram showing forces and moments in the horizontal plane.

The appropriate dimensionless variables are

$$\beta = u/L ; \quad u = L\beta \quad (33)$$

$$B_4 = H/P ; \quad H = PB_4 = B_4P \quad (34)$$

$$n = N/PL ; \quad N = PLn \quad (35)$$

Dimensionless shear  $B_4$  is one of ultimately eight unknown scalar quantities ( $B_1, B_2, \dots, B_8$ ) which will be introduced in the process of determining  $Q$ . The alternate forms indicated in the preceding equations are intended to familiarize the reader with notational forms which will be freely employed in later developments.

The illustration of the rotated cross section in figure 2 (page 10) shows that the moment causing bending about the  $\eta$  axis is  $N \cos \phi - M \sin \phi$ , which reduces to  $N - \phi M$  by small angle approximation. From elementary bending theory we have

$$EI_2 u'' = N - \phi M \quad (36)$$

and by substitution from equation 7 it may be written as

$$\ddot{\beta} = Qx^2(n - \phi x^9) \quad (37)$$

Examination of the simple statics of figure 4 (page 18) yields

$$Hz + N(0) = N(z) \quad (38)$$

In dimensionless form this becomes

$$B_4\zeta + n(0) = n(\zeta) \quad (39)$$

and may be rewritten as

$$n = B_4px^7 + B_5x^7 \quad (40)$$

$$n(0) = B_5 \quad (41)$$

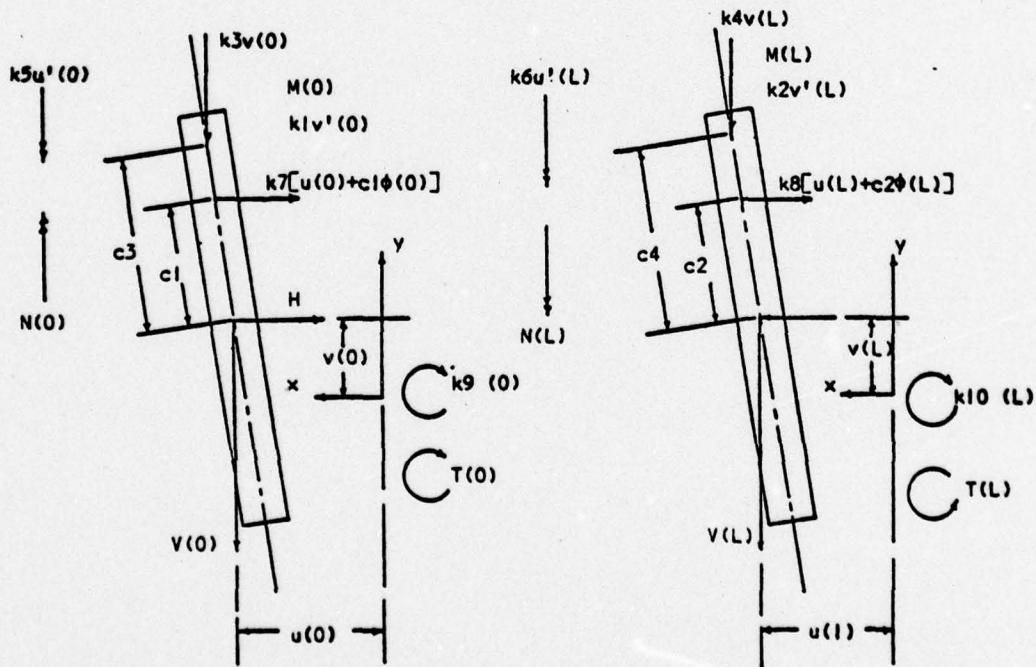


Figure 5. Force and moment diagrams of end sections.



Figures 5 and 6 (page 20) show the forces and moments used to develop the torsion moment equation

$$T(z) = T(0) - H[v(z) - v(0)] + V(z)u(z) - V(0)u(0)$$

$$+ \int_0^z w(z)[u(z) + e(z)\phi(z)]dz \quad (42)$$

Introducing the dimensionless function

$$t = T/PL \quad (43)$$

and using additional dimensionless functions previously defined, equation 31 may be written as

$$t = t(0) - QB4[x10 - x10(0)] + x8\beta - x8(0)\beta(0) + px5(\beta + x6\phi) \quad (44)$$

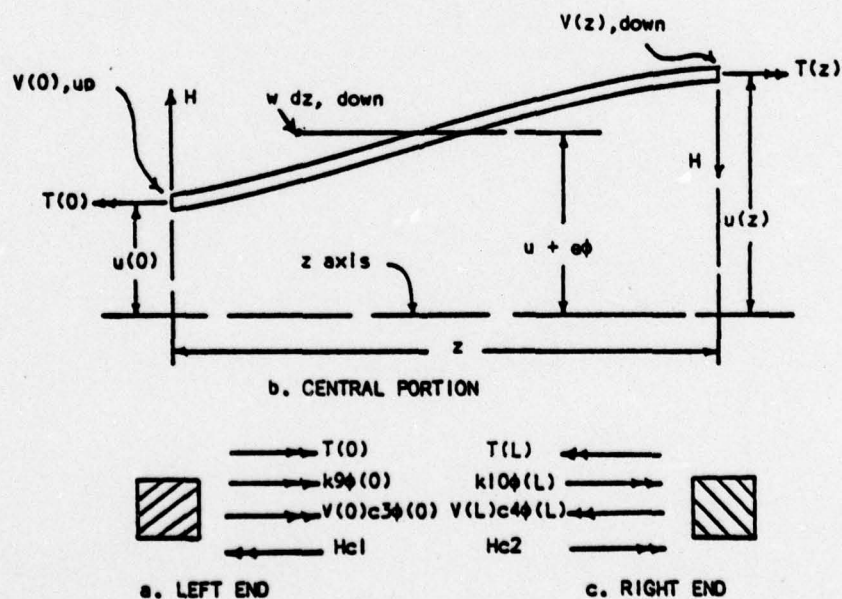


Figure 6. Diagram of left, central, and right portions for torsion analysis.

Reference (1) develops the following third order equation which accounts for the warping rigidity of the beam.

$$C_1 \phi''' = C \phi' + T - u'M \quad (45)$$

In dimensionless form this may be written as

$$\ddot{\phi} = x_4(Q_t - Qx_9\dot{\beta} + x_3\dot{\phi}) \quad (46)$$

The function  $x_4$  has previously been defined as

$$x_4 = RL^2/C_1 \quad (47)$$

$C_1$  is the coefficient of the highest (i.e., third) order derivative in equation 45, and if  $C_1$  becomes small, very troublesome mathematical difficulties are encountered, cf. Reference (3). Accordingly, the analysis for the case  $C_1 \equiv 0$  cannot be obtained from that for  $C_1 \neq 0$ , given in reference (1), except with the greatest of difficulty. It is for this reason that the separate analysis represented by this thesis, was undertaken. The analysis herein, to this point, is, as has been said before, essentially identical to that in reference (1). From this point on there are essential differences.

The present analysis presumes that  $C_1 = 0$ , so that equation 35 takes the form

$$\dot{\phi} = Qx_3^{-1}(x_9\dot{\beta} - t) \quad (48)$$

The inverse of  $x_3$ ,  $x_3^{-1}$ , occurs frequently in what follows. The term is clumsy in this form and will be redesignated as

$$Y = x_3^{-1} \quad (49)$$

Therefore equation 48 is written as

$$\dot{\phi} = QY(x_9\dot{\beta} - t) \quad (50)$$

Two of the end constraint conditions of the original problem, namely those relating to warping constraints at the end, are not available for and do not appear in the present specialized problem.

The original problem had six constraints at each end. The present problem has five constraints at each end. Four constraint conditions were employed in solving the problem of bending in the vertical plane. Thus a total of six constraint conditions remain at this stage of the development. The first four result from the constraints against horizontal deflection and end rotations in the x-z plane. The last two result from constraints against rotation about the z axis.

$$a_5\dot{\beta}(0) - Q(1-a_5) n(0) = 0 \quad (51)$$

$$a_6\dot{\beta}(1) + Q(1-a_6) n(1) = 0 \quad (52)$$

$$a_7[\beta(0)+G_1\phi(0)] + QB_4(1-a_7) = 0 \quad (53)$$

$$a_8[\beta(1)+G_2\phi(1)] - QB_4(1-a_8) = 0 \quad (54)$$

$$a_9\phi(0)+Q(1-a_9)[t(0)+G_3x_8(0)\phi(0)-B_4G_1] = 0 \quad (55)$$

$$a_{10}\phi(1)-Q(1-a_{10})[t(1)+G_4x_8(1)\phi(1)-B_4G_2] = 0 \quad (56)$$

Although this notation appears formidable, it is convenient, explicit, and unequivocal. The terms are



sufficiently complicated that no matter what notation might be used careful attention to detail is required.

There is no need to demonstrate the complete development of the six boundary equations, but for illustrative purposes the derivations of two typical constraint equations, 51 and 53, will be given.

The constraint against rotation of the left end about a vertical (y) axis, as shown in figure 6 (page 20) results in the following equation.

$$k_5 u'(0) - N(0) = 0 \quad (57)$$

Employing the dimensionless variable  $\beta = u/L$  and  $n = N/PL$ , using equation 17, writing the spring constant  $k_5$  in terms of the spring parameter  $a_5$ , viz,

$$k_5 = Ra_5/L(1-a_5) \quad (58)$$

and recalling equation 5, it is easy to obtain the equation

$$a_5 \dot{\beta}(0) - Q(1-a_5)n(0) = 0 \quad (59)$$

This is equation 51.

Equation 53 defines the constraint against horizontal deflection of the left end of the beam. The deflection is  $u(0) + c_1 \phi(0)$ , see figure 5 (page 19), where the second term results from permitting the spring to be attached at a distance  $c_1$  above the centroidal axis. The constraint equation is

$$k_7 [u(0) + c_1 \phi(0)] + H = 0 \quad (60)$$

where as previously defined  $H = PB_4$ ,  $c_1 = LG_1$ , and

$$k_7 = a_7 R / L^3 (1 - a_7) \quad (61)$$

Using the dimensionless quantities previously introduced, it is a simple matter to put equation 60 in the form exhibited as equation 53.

The structural analysis has now been completed and is represented by equations 37, 44, 50, 51, 52, 53, 54, 55, and 56. The mathematical method of solution will be addressed next.

#### IV. MATHEMATICAL ANALYSIS

Before going into the details of the analysis, we will try to describe it in general terms.

An expression is formed for  $\dot{\phi}$  which involves an unknown function ( $x_{11}$  in what follows) the initial form of which is arbitrarily assumed except for the normalizations

$$x_{11}(0) = 0 \quad (62)$$

$$*p|x_{11}| = 1 \quad (63)$$

The corresponding function  $\phi$  is obtained by integration and equation 37 is used to get  $\ddot{\beta}$ . This is followed by two more integrations to get  $\dot{\beta}$  and  $\beta$ . These are employed in the torsion equation 44 to get the function  $t$ . Finally this is used in equation 48 to get a new expression for  $\dot{\phi}$ . Each integration introduces a new unknown scalar quantity. In one way or another a total of eight such unknown scalar constants  $B_1, B_2, \dots, B_8$  are introduced. The six constraint equations, plus two more which will be introduced in what follows, provide a set of eight linear homogeneous equations in these unknown  $B$ 's which thus leads to an eigenproblem of eighth order. The parameter  $Q$  appears in the coefficients to the first and second powers. The quadratic eighth order eigenproblem is transformed to a linear sixteenth order eigenproblem. Since, generally, both zero and infinite eigenvalues are contained in the solution, a special, relatively new



algorithm, the QZ method, capable of dealing with such systems, is employed for the solution of this eigenproblem. The appropriate eigenvalue is chosen and the corresponding eigenvector provides the solution for the unknown B's so that the new  $x_{11}$  can be uniquely determined. This new  $x_{11}$  is used in place of the original  $x_{11}$  and the process is repeated until there is convergence. No proof is offered that indeed convergence must ultimately take place; because of the complexity of the problem such a proof might be very difficult to establish. However, experience with the procedure described here indicates that convergence may indeed be expected. The details of this procedure are now given in what follows.

The function  $\dot{\phi}$  is initially specified as

$$\dot{\phi} = B_1 x_{11} + B_2 x_7 \quad (64)$$

where  $x_{11}$  is an arbitrarily assumed function satisfying equations 62 and 63 and where  $B_1$  and  $B_2$  are the third and fourth of eight unknown constants that will be employed. The first was  $B_4$  from equation 34 and the second was  $B_5$  from equation 41.

Integrating equation 64 yields

$$\phi = B_1 p x_{11} + B_2 p x_7 + B_3 x_7 \quad (65)$$

and when equations 40 and 65 are substituted into equation 37 we arrive at

$$\ddot{\beta} = -B1Qx2x9px11 - B2Qx2x9px7 - B3Qx2x9 + B4Qx2px7 + B5Qx2 \quad (66)$$

Integrating twice more, we get

$$\dot{\beta} = -B1Qpx2x9px11 - B2Qpx2x9px7 - B3Qpx2x9 + B4Qpx2px7 + B5Qpx2 + B6Qx7 \quad (67)$$

$$\beta = -B1Qppx2x9px11 - B2Qppx2x9px7 - B3Qppx2x9 + B4Qppx2px7 + B5Qppx2 + B6Qpx7 + B7Qx7 \quad (68)$$

For convenience the next step is to rewrite equation 44 as

$$t = q\beta + px5x6\phi - QB4X - Qx8(0)B7 + QB8 \quad (69)$$

$$\text{where} \quad t(0) = QB8 \quad (70)$$

$$X = x10 - x10(0) \quad (71)$$

$$q = px5 + x8 \quad (72)$$

The symbol  $q$  denotes an operator acting on the function  $\beta$ , not a function which is multiplied by  $\beta$ .

When the preceding equations, evaluated at  $\zeta=0$  or  $\zeta=1$  as the case may be, are substituted in boundary conditions equations 51 through 56, a system of six linear simultaneous homogeneous equations in eight unknown constants  $B$  is obtained.

In the previous section, which addressed the structural analysis, equations 51 and 53 were derived for illustrative purposes. Now for illustrative continuity these two equations will be used to show the method of formulation of the six linear simultaneous homogeneous equations.

Recalling that equation 51 is

$$a_5 \dot{\beta}(0) - Q(1-a_5) n(0) = 0$$

and substituting

$$\dot{\beta}(0) = QB_6 \quad (73)$$

$$n(0) = B_5 \quad (74)$$

we now factor out Q leaving

$$-B_5(1-a_5) + B_6 a_5 = 0 \quad (75)$$

The second illustrative equation, 53, is

$$a_7[\beta(0) + G_1 \phi(0)] + QB_4(1-a_7) = 0$$

and upon substituting

$$\phi(0) = B_3 \quad (76)$$

$$\beta(0) = B_7 Q \quad (77)$$

we arrive at

$$B_3 G_1 a_7 + B_4 Q(1-a_7) + B_7 Q a_7 = 0 \quad (78)$$

If equations 67 and 69 are substituted into equation 50 the following "new"  $\dot{\phi}$  is created.



$$\begin{aligned}
\dot{\phi} = & Y[B1(-Q^2 x_9 p_{x2} x_9 p_{x11} + Q^2 q_{ppx2} x_9 p_{x11} - Q p_{x5} x_6 p_{x11}) \\
& + B2(-Q^2 x_9 p_{x2} x_9 p_{x7} + Q^2 q_{ppx2} x_9 p_{x7} - Q p_{x5} x_6 p_{x7}) \\
& + B3(-Q^2 x_9 p_{x2} x_9 + Q^2 q_{ppx2} x_9 - Q p_{x5} x_6) \\
& + B4(Q^2 x_9 p_{x2} p_{x7} + Q^2 X - Q^2 q_{ppx2} p_{x7}) \\
& + B5(Q^2 x_9 p_{x2} - Q^2 q_{ppx2}) \\
& + B6(Q^2 x_9 - Q^2 q_{px7}) \\
& + B8(-Q^2)] \tag{79}
\end{aligned}$$

(The coefficient of B7 vanishes identically.)

To complete the eigensystem involving the eight unknown B's, two additional equations are needed.

The seventh equation, consistent with equation 62, is

$$\text{new } \dot{\phi}(0) - B2 = 0 \tag{80}$$

The eighth equation is

$$(*p\dot{\phi})_{\text{old}} - (*p\dot{\phi})_{\text{new}} = 0 \tag{81}$$

and is arrived at by observing that if  $\dot{\phi}$  given in equation 64 and created from an assumed  $x_{11}$ , were, somehow, a "correct"  $\dot{\phi}$  then it and the  $\dot{\phi}$  recovered in equation 79 would be identically equal to each other. The integration in equation 81 achieves a certain "smoothing" or averaging which has been found to be preferable to simply equating the new and old functions at an arbitrary value of  $\zeta$ .

The coefficients of the eigensystem equations generally involve  $Q$  to the zeroth, first, and second powers so that the system may be exhibited in the form

$$(D + EQ + FQ^2)B = 0 \tag{82}$$

The nonzero elements of matrices D, E, and F are

$D_{15} = a_5 - 1, D_{16} = a_5, D_{21} = -a_6 * p_{x2} * x_9 * p_{x11},$   
 $D_{22} = -a_6 * p_{x2} * x_9 * p_{x7}, D_{23} = -a_6 * p_{x2} * x_9, D_{24} = a_6 * p_{x2} * p_{x7} - a_6 + 1,$   
 $D_{25} = a_6 * p_{x2} - a_6 + 1, D_{26} = a_6, D_{33} = G_{1a7},$   
 $D_{41} = a_8 G_2 * p_{x11}, D_{42} = a_8 G_2, D_{43} = a_8 G_2,$   
 $D_{53} = a_9, D_{61} = a_{10} * p_{x11}, D_{62} = a_{10}, D_{63} = a_{10},$   
 $D_{72} = -1, D_{81} = - * p_{x11}, D_{82} = -1, E_{34} = 1 - a_7,$   
 $E_{37} = a_7, E_{41} = -a_8 * p_{px2} * x_9 * p_{x11},$   
 $E_{42} = -a_8 * p_{px2} * x_9 * p_{x7}, E_{43} = -a_8 * p_{px2} * x_9,$   
 $E_{44} = a_8 * p_{px2} * p_{x7} + a_8 - 1, E_{45} = a_8 * p_{px2},$   
 $E_{46} = a_8, E_{47} = a_8, E_{53} = (1 - a_9) G_3 * x_8(0),$   
 $E_{54} = (a_9 - 1) G_1, E_{61} = (a_{10} - 1) [* p_{x5} * x_6 * p_{x11} + G_4 * x_8(1) * p_{x11}],$   
 $E_{62} = (a_{10} - 1) [* p_{x5} * x_6 * p_{x7} + G_4 * x_8(1)],$   
 $E_{63} = (a_{10} - 1) [* p_{x5} * x_6 + G_4 * x_8(1)], E_{64} = (1 - a_{10}) G_2,$   
 $E_{81} = - * p_{Ypx5} * x_6 * p_{x11}, E_{82} = - * p_{Ypx5} * x_6 * p_{x7},$   
 $E_{83} = - * p_{Ypx5} * x_6, F_{58} = 1 - a_9,$   
 $F_{61} = (1 - a_{10}) * q_{ppx2} * x_9 * p_{x11}, F_{62} = (1 - a_{10}) * q_{ppx2} * x_9 * p_{x7},$   
 $F_{63} = (1 - a_{10}) * q_{ppx2} * x_9, F_{64} = (1 - a_{10}) (* X - * q_{ppx2} * p_{x7}),$   
 $F_{65} = (a_{10} - 1) * q_{ppx2}, F_{66} = (a_{10} - 1) * q_{px7},$   
 $F_{68} = a_{10} - 1, F_{76} = x_9(0) Y(0), F_{78} = -Y(0),$   
 $F_{81} = - * p_{Yx9} * p_{x2} * x_9 * p_{x11} + * p_{Yqppx2} * x_9 * p_{x11},$   
 $F_{82} = - * p_{Yx9} * p_{x2} * x_9 * p_{x7} + * p_{Yqppx2} * x_9 * p_{x7},$   
 $F_{83} = - * p_{Yx9} * p_{x2} * x_9 + * p_{Yqppx2} * x_9,$   
 $F_{84} = * p_{Yx9} * p_{x2} * p_{x7} + * p_{YX} - * p_{Yqppx2} * p_{x7},$   
 $F_{85} = * p_{Yx9} * p_{x2} - * p_{Yqppx2},$   
 $F_{86} = * p_{Yx9} - * p_{Yqpx7},$   
 $F_{88} = - * p_Y$

[For reference purposes all the preceding equations bear the number 84.]

This eighth order quadratic eigensystem may be expanded to a sixteenth order linear eigensystem in the form

$$VZ = QWZ \quad (85)$$

where

$$V = \begin{bmatrix} E & D \\ I & O \end{bmatrix} \quad (86)$$

$$W = \begin{bmatrix} -F & O \\ O & I \end{bmatrix} \quad (87)$$

$$Z = \begin{bmatrix} QB \\ B \end{bmatrix} \quad (88)$$

and I and O respectively represent the eighth order unit and null matrices.

The eigenproblem has now been defined in a linear form, but because the matrices V and W may be singular not just any method will satisfactorily solve the problem. We have chosen the QZ method described in reference (3). From the sixteen eigenvalues produced the smallest positive eigenvalue is selected. The last eight elements of the associated eigenvector are the desired constants B1 through B8.

Using equation 79 a new  $\dot{\phi}$  is now formed and from it a new x11 is created by means of the following normalization process.

$$x11_{new} = [\dot{\phi}_{new} - \dot{\phi}_{new}^{(0)}] / [*p|\dot{\phi}_{new} - \dot{\phi}_{new}^{(0)}|] \quad (89)$$



The iteration process is now repeated until satisfactory convergence is attained. The converged smallest positive eigenvalue  $Q$  is now used to determine the buckling load from either equation 3 or equation 6.

## V. COMPUTER IMPLEMENTATION OF THEORY

To this point the functions  $x$  have been treated as continuous functions for use in an analytic solution. However, it is obvious that solving the problem, even for simple functions, by analytic means is prohibitively difficult. To overcome this problem a numerical method of solution was employed. The beam was divided into a finite number of equal sections. The functions  $x$  were then redefined as vectors with Fortran designations  $x(1,I)$ ,  $x(2,I)$ ..., etc. where  $I = 1, 2, 3, \dots, N$ . The elements of any such vector represent function values at the  $N$  points of subdivision of the beam into equal subsections of length  $L/(N-1)$ . The vector  $x(10,I)$  was used twice, first to represent  $x_{10}$  and then to store  $X$  of equation 71.

The vectors in this form readily lend themselves to manipulation. The simple subroutines developed to perform the operations are, with the exception of the integration scheme, not worthy of note here.

Trapezoidal integration was first used during the debugging of the raw, untested program. This method gives satisfactory results with an error of  $O(h^2)$ , where  $h$  is the interval length. Once it was established that the program would give satisfactory solutions the refinement process was begun. A higher order numerical integration scheme was employed next. It is described by Milne in reference (5),

and results in an error  $O(h^5)$ .

The QZ subroutine is a canned program in the library of the Naval Postgraduate School Computer Center. It was transcribed from reference (6), with very little change, by Mr. Roger Hilleary. To date, it has been the only program that has given satisfactory solutions to the present eigenvalue problem.

The complete theory described in the preceding sections of this thesis has been implemented in the form of a subroutine LATBRO, a complete listing of which is contained in Appendix A.

The user must write a main program which supplies dimensioning statements, number of intervals, specifications for vectors  $x_1$  through  $x_6$ , specifications for the constants  $G$ , and the ten spring parameters. The following is an example of such a main program. It was used to solve what is listed as Case 6, Table 1, page 37.

```
IMPLICIT REAL*8(A-H,O-Z)
REAL*8 X(30,101),ALFA(12),G(4),S,ONE,ZERO
INTEGER KP(10)
COMMON X,ALFA,G,N,KP
N=101
ZERO=0.D+0
ONE=1.D+0
DO 1 I=1,N
X(1,I)=ONE
X(2,I)=ONE
X(3,I)=ONE
X(5,I)=ZERO
X(6,I)=ZERO
1 CONTINUE
X(5,1)=ONE
DO 2 I=1,4
2 G(I)=ZERO
DO 3 I=1,12
```



```
3 ALFA(1)=ZERO
  ALFA(2)=ONE
  ALFA(4)=ONE
  ALFA(6)=ONE
  ALFA(8)=ONE
  ALFA(10)=ONE
  KP(3)=1
  CALL LATBRO(S)
  STOP
  END
```

## VI. TESTING AND VERIFICATION

As mentioned in the previous section, the numerical integration methods that have been employed have their shortcomings.

If the integrand is "smooth," either the trapezoidal or the higher order Milne method may be used. The latter is preferable because of its greater accuracy.

Both methods introduce errors, however, when there are steps, impulses, or doublets to be integrated. We have found that the Milne and trapezoidal methods can be used to solve problems with a concentrated load. The error depends linearly on the reciprocal of the number of elements into which the beam is divided and extrapolation may be used to obtain an excellent result. As yet we do not have experience with concentrated moment loadings (doublets). This subject is discussed in more detail in reference (1).

The following table shows the cases that have been tested to date. In each of these cases all the G's and the eccentricity  $x_6$  were zero. Each of these has been successful.

The values for Q shown in the last column of the table were obtained with  $x_2 \equiv 1$ ,  $x_3 \equiv 1$ , and by simple scaling analysis lead to total load

$$P = Q \sqrt{CEI_2/L^2} \quad (89)$$

TABLE I  
Cases of Uniform Beams Analyzed by Subroutine LATBRO  
(x6 = 0, G1 = G2 = G3 = G4 = 0)

Case Number	a1	a2	a3	a4	a5	a6	a7	a8	a9	a10	Loading Type	Q
1	0	1	0	1	0	1	0	1	0	1	UNIF	12.854
2	1	0	1	0	1	0	1	0	1	0	UNIF	12.854
3	0	0	1	1	1	1	1	1	1	1	UNIF	47.58
4	1	1	1	1	0	0	1	1	1	1	UNIF	97.11
5	0	0	1	1	0	0	1	1	1	1	CONC MIDSPAN	16.936
6	0	1	0	1	0	1	0	1	0	1	CONC LEFT END	4.018



for lateral buckling. The values reported agree with values to be found in reference (2).

The following are details of several cases.

Case 1

As shown, it is a cantilever beam fully restrained at the right end. The loading is uniform.

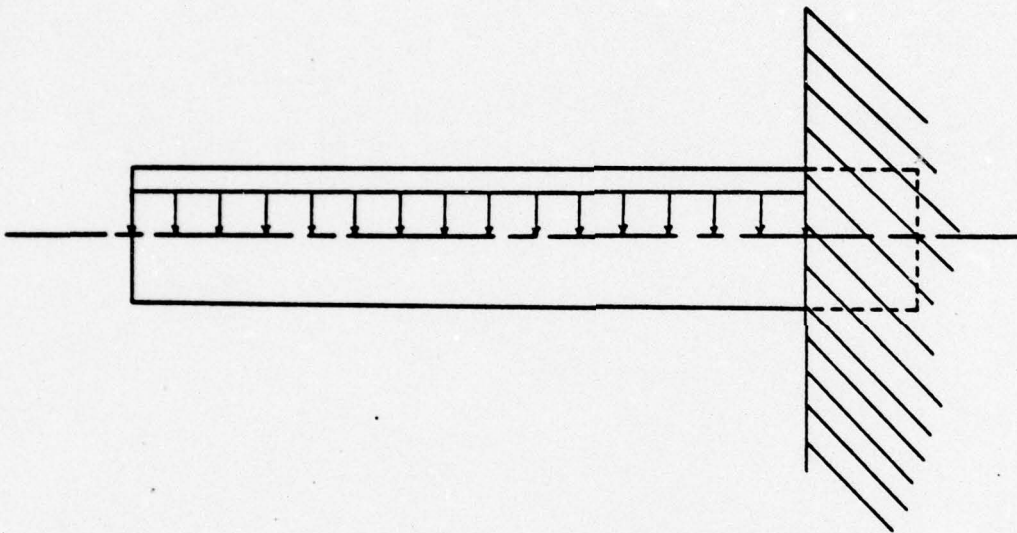


Figure 7. Illustration of side view of Case 1 beam.

The subroutine produced a converged value of  $Q$  of 12.854. Timoshenko's result is 12.85; cf. page 261 ref (2).

Case 2

This is simply Case 1 turned end for end. It was used to test additional equations and also resulted in a converged critical load output of 12.854 which is identical to the result for Case 1, as it should be.

Case 5

The converged critical load output was 16.936 and Timoshenko's result is 16.94; cf. page 269 ref (2).

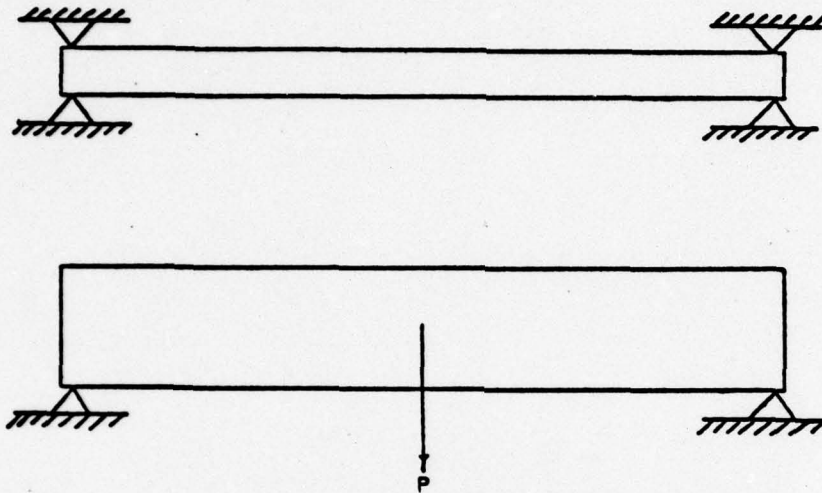


Figure 8. Top and side views Case 5 beam.

In several of the cases listed in the table above, the loading does not interact with one or more of the springs, so that the corresponding spring parameters could have an arbitrary value without changing the results.

## VII. CONCLUSIONS AND RECOMMENDATIONS

The subroutine LATBRO in its present form should be able to solve beam problems with beams that have uniform or variable cross sections, with loading on the centroidal axis, and with spring constraints attached at the centroidal axis. Loading that involves concentrated moments, as mentioned before, has not yet been tested.

The program was designed to solve problems where loading and spring constraints may act away from the centroidal axis. It is recommended that follow on testing begin with

1. loading off the centroidal axis.
2. spring constraints attached away from the centroidal axis.
3. concentrated moment loading.

It is expected, with the exception of minor programming errors, that problems with any combination of the above will be readily solved by the subroutine.



## Listing of LATBRO and Adjunct Subroutines

```

SUBROUTINE LATBRO(S)
IMPLICIT REAL*8(A-H,O-Z)
REAL*8 X(30,101),ALFA(12),G(4),S
REAL*8 WKAREA(800),ZERO,ONE,ONEG,DELTA,LAM
REAL*8 C(4,4),CC(4,4),D(8,8),E(8,8),F(8,8),B(8),DEN(16),EN
REAL*8 AA(16,16),BB(16,16)
REAL*8 EKS(16,16),NUMR(16),NUMI(16)
REAL*8 RESULT(50)
INTEGER ITER(16)
LOGICAL WANTX
COMMON X,ALFA,G,N,KP
KKG=1
ZERO=0.D+0
ONE=1.D+0
ONEG=-ONE
DO 1 I=1,N
1 X(7,I)=ONE
** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** **
THE FOLLOWING PRINTS ADVICE TO THE USER THAT HE HAS CHOSEN THE
FIRST ORDER CASE WHICH MAKES NO CONSIDERATION OF WARPING.
** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** **
WRITE(6,2004)
2004 FORMAT('O: USER NOTE: YOU HAVE CHOSEN THE USE OF THE FIRST ORDER
C SUBROUTINE AND NO CONSIDERATION WILL BE GIVEN TO THE WARPING PHEN
C OMENON.')
```





```

IDGT=0
CALL LEQTI1F(C,M,NA,NA,CC,IDGT,WKAREA,IER)
* * * * *
VECTOR CC NOW CONTAINS (DIMENSIONLESS) SHEAR, MOMENT,
SLOPE, AND DEFLECTION AT THE LEFT END.
* * * * *
CALL SUBS(13,20,CC(1))
CALL MULS(20,8,ONEG)
CALL INTV(8,9)
CALL ADDS(9,9,CC(2))
CALL MULV(1,9,10)
CALL INTV(10,11)
CALL ADDS(11,11,CC(3))
CALL INTV(11,10)
CALL ADDS(10,10,CC(4))
CALL PRIV(8,2,8)
CALL PRIV(8,2,9)
CALL PRIV(9,2,9)
CALL PRIV(10,2,10)
CALL PRIV(10,2,10)
* * * * *
VECTORS 8 (SHEAR), 9 (MOMENT), AND 10 (DEFLECTION) HAVE BEEN
DETERMINED. NOW WE ARE READY TO SOLVE THE LATERAL BUCKLING
PROBLEM. FIRST WE ZERO D(8,8), E(8,8), AND F(8,8).
* * * * *
DC 10 I=1,8
DO 10 J=1,8
D(I,J)=ZERO
E(I,J)=ZERO
F(I,J)=ZERO
10
* * * * *
NOW WE CALCULATE THOSE COEFFICIENTS WHICH DO NOT DEPEND ON X11.
THE FIRST COMMAND BELOW TURNS X10 UNTO CAPITAL X10.
* * * * *
CALL SUBS(10,10,X(10,1))

```

```

BRK11240
BRK11250
BRK11260
BRK11270
BRK11280
BRK11290
BRK11300
BRK11310
BRK11320
BRK11330
BRK11340
BRK11350
BRK11360
BRK11370
BRK11380
BRK11390
BRK11400
BRK11410
BRK11420
BRK11430
BRK11440
BRK11450
BRK11460
BRK11470
BRK11480
BRK11490
BRK11500
BRK11510
BRK11520
BRK11530
BRK11540
BRK11550
BRK11560
BRK11570
BRK11580
BRK11590
BRK11600
BRK11610
BRK11620
BRK11630
BRK11640
BRK11650
BRK11660
BRK11670
BRK11680
BRK11690
BRK11700
BRK11710

```



```

D(1,5)=ALFA(5)-ONE
DD(1,6)=ALFA(5)
DD(2,6)=ALFA(6)
DD(3,6)=ALFA(7)*G(1)
EE(3,4)=ONE-ALFA(7)
EE(3,7)=ALFA(7)
EE(4,3)=ALFA(8)*G(2)
DE(4,2)=ALFA(8)*G(2)
EE(4,6)=ALFA(8)
EE(4,7)=ALFA(8)
EE(5,3)=(ONE-ALFA(8))*G(3)*X(8,1)
DE(5,3)=ALFA(9)
DE(5,4)=(ALFA(9)-ONE)*G(1)
FE(5,8)=ONE-ALFA(9)
DE(6,2)=ALFA(10)
DE(6,3)=ALFA(10)
FE(6,4)=(ONE-ALFA(10))*G(2)
DE(7,2)=ALFA(10)-ONE
DD(8,2)=ONEG
AL(7,8)=X(14,1)
FF(7,6)=X(14,1)*X(9,1)
CALL INTV(14,15)
CALL(8,8)=-X(15,N)
CALL INTV(7,12)
CALL MULV(9,12,13)
CALL MULV(2,13,20)
CALL INTV(20,16)
DI(2,2)=-ALFA(6)*X(19,N)
CALL INTV(19,17)
CALL QVEC(17,18)
E(4,2)=-ALFA(8)*X(17,N)
FE(6,2)=(ONE-ALFA(10))*X(18,N)
CALL MULV(2,9,14)
CALL INTV(14,15)
DI(2,3)=-ALFA(6)*X(15,N)
CALL INTV(15,16)
E(4,3)=-ALFA(8)*X(16,N)
CALL QVEC(16,17)
FE(6,3)=(ONE-ALFA(10))*X(17,N)
CALL MULV(2,12,14)
CALL INTV(14,15)
DI(2,4)=ALFA(6)*X(15,N)-ALFA(6)+ONE
CALL INTV(15,16)
E(4,4)=ALFA(8)*X(16,N)+ALFA(8)-ONE
CALL INTV(2,14)
DI(2,5)=ALFA(6)*X(14,N)-ALFA(6)+ONE

```









```

C * * * * * CALL INTV(7,11)
C 30 CONTINUE
CALL PRIV(1,2,11)
CALL PRIV(1,2,11)
CALL INTV(1,1,12)
CALL MULV(9,12,13)
CALL MULV(2,13,14)
CALL INTV(1,4,15)
D(2,1)=-ALFA(6)*X(15,N)
CALL INTV(15,16)
E(4,1)=-ALFA(8)*X(16,N)
D(4,1)=ALFA(8)*G(2)*X(12,N)
D(6,1)=ALFA(10)*X(12,N)
CALL QVEC(16,17)
F(6,1)=(ONE-ALFA(10))*X(17,N)
D(8,1)=-X(12,N)
CALL MULV(9,15,18)
CALL SUBV(17,18,19)
CALL DIVV(19,3,20)
CALL INTV(20,13)
F(8,1)=X(13,N)
CALL MULV(6,12,13)
CALL MULV(5,13,14)
CALL INTV(14,15)
S=X(8,N)
CALL MULS(12,16,5)
E(6,1)=(ALFA(10)-ONE)*(X(15,N)+(G(4)*X(16,N)))
CALL DIVV(15,3,17)
CALL INTV(17,18)
E(8,1)=-X(18,N)
F(8,1)=X(18,N)
29 WRITE(6,29)
DC 500 I=1,8
WRITE(6,29)
WRITE(6,29)
DO 500 J=1,8
WRITE(6,501) D(I,J),E(I,J),F(I,J)
500 CONTINUE
501 F(6,29)=1P3E20.8)
DC 500 I=1,8
WRITE(6,29)
WRITE(6,29)
DC 500 J=1,8
WRITE(6,501) D(I,J),E(I,J),F(I,J)
500 CONTINUE
CCCCCCC

```

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BRK13160
BRK13170
BRK13180
BRK13190
BRK13200
BRK13210
BRK13220
BRK13230
BRK13240
BRK13250
BRK13260
BRK13270
BRK13280
BRK13290
BRK13300
BRK13310
BRK13320
BRK13330
BRK13340
BRK13350
BRK13360
BRK13370
BRK13380
BRK13390
BRK13400
BRK13410
BRK13420
BRK13430
BRK13440
BRK13450
BRK13460
BRK13470
BRK13480
BRK13490
BRK13500
BRK13510
BRK13520
BRK13530
BRK13540
BRK13550
BRK13560
BRK13570
BRK13580
BRK13590
BRK13600
BRK13610
BRK13620
BRK13630

```





```

X(20,1)=ZERC
IF(DABS(DEN(1))) LT.1 D-20) GO TO 42
IF(DABS(NUMR(1))) LT.1 D-20) GO TO 42
IF(DABS(NUMI(1))) LT.1 D-20) GO TO 42
IF(KP(3).EQ.1) X(19,1)=DEN(1)/NUMR(1)
IF(KP(3).EQ.2) X(19,1)=NUMR(1)/DEN(1)
IF(X(19,1).GT.ZERO) X(20,1)=ONE
KP(9)=7
IF(KP(9).EQ.7) WRITE(6,41) I, X(19,1), X(20,1), NUMR(1), DEN(1)
41 FORMAT(5X,15,1P7E16.4)
42 CONTINUE
KLEE=0
NVEC=0
S=ZERO
DO 43 I=1,16
IF(X(20,1)) LT.5 D-1) GO TO 43
IF(KLEE.EQ.0) S=X(19,1)
IF(KLEE.EQ.0) NVEC=I
KLEE=1
IF(X(19,1).LT.S) NVEC=I
IF(X(19,1).LT.S) S=X(19,1)
43 CONTINUE
RESULT(KKK)=S
WRITE(6,29) NVEC,S
44 FORMAT(5X,'WE SELECT EIGVAL NUMBER ',I2,' WHICH EQUALS ',D20.5)
45 WRITE(6,29)
WRITE(6,45)
46 FORMAT(5X,'THE CORRESPONDING EIGVEC (WE USE ONLY REAL PART) IS:')
47 WRITE(6,29)
DO 46 I=1,8
IP=I+8
B(I)=EKS(IP,NVEC)
48 WRITE(6,47) I,B(I)
49 CONTINUE
50 FORMAT(10X,15,2D19.6)
51 LAM=S
52 CONTINUE
53 WRITE(6,412)
54 THE FOLLOWING IS WRITE(6,2002) CPUT.
55 2002 FORMAT(3X,316,1P1E19.5)
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BRK14120
BRK14130
BRK14140
BRK14150
BRK14160
BRK14170
BRK14180
BRK14190
BRK14200
BRK14210
BRK14220
BRK14230
BRK14240
BRK14250
BRK14260
BRK14270
BRK14280
BRK14290
BRK14300
BRK14310
BRK14320
BRK14330
BRK14340
BRK14350
BRK14360
BRK14370
BRK14380
BRK14390
BRK14400
BRK14410
BRK14420
BRK14430
BRK14440
BRK14450
BRK14460
BRK14470
BRK14480
BRK14490
BRK14500
BRK14510
BRK14520
BRK14530
BRK14540
BRK14550
BRK14560
BRK14570
BRK14580
BRK14590

```





```

CALL ADDV(12,19,13)
CALL DUPV(13,12)
* * * * *
NOW B3 PORTION.
* * * * *
CALL MULV(2,9,13)
CALL INTV(13,14)
CALL INTV(14,15)
CALL QVEC(15,16)
CALL MULV(9,14,17)
CALL SUBV(16,17,18)
S=LAM#2
CALL MULS(18,19,S)
CALL MULV(5,6,13)
CALL INTV(13,14)
S=LAM
CALL MULS(14,15,S)
CALL SUBV(19,15,20)
S=B(3)
CALL MULS(20,19,S)
CALL ADDV(12,19,13)
CALL DUPV(13,12)
* * * * *
NCW B4 PORTION.
* * * * *
CALL INTV(7,13)
CALL MULV(2,13,14)
CALL INTV(14,15)
CALL INTV(15,16)
CALL QVEC(16,17)
CALL MULV(9,15,18)
CALL ADDV(18,10,19)
CALL SUBV(19,17,20)
S=LAM#2
CALL MULS(20,13,S)
S=B(4)
CALL MULS(13,14,S)
CALL ADDV(12,14,15)
CALL DUPV(15,12)
* * * * *

```

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BRK15080
BRK15090
BRK15100
BRK15110
BRK15120
BRK15130
BRK15140
BRK15150
BRK15160
BRK15170
BRK15180
BRK15190
BRK15200
BRK15210
BRK15220
BRK15230
BRK15240
BRK15250
BRK15260
BRK15270
BRK15280
BRK15290
BRK15300
BRK15310
BRK15320
BRK15330
BRK15340
BRK15350
BRK15360
BRK15370
BRK15380
BRK15390
BRK15400
BRK15410
BRK15420
BRK15430
BRK15440
BRK15450
BRK15460
BRK15470
BRK15480
BRK15490
BRK15500
BRK15510
BRK15520
BRK15530
BRK15540
BRK15550

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* * * * * NGW B5 PORTION.
* * * * *
CALL INTV(2,13)
CALL INTV(13,14)
CALL QVEC(14,15)
CALL MULV(9,13,16)
CALL SUBV(16,15,17)
S=LAM**2
CALL MULS(17,18,S)
S=B(5)
CALL MULS(18,19,S)
CALL ADDV(12,19,20)
CALL DUPV(20,12)
* * * * *
* * * * * NCH B6 PORTION.
* * * * *
CALL INTV(7,13)
CALL QVEC(13,14)
CALL SUBV(9,14,15)
S=LAM**2
CALL MULS(15,16,S)
S=B(6)
CALL MULS(16,17,S)
CALL ADDV(12,17,18)
CALL DUPV(18,12)
* * * * *
* * * * * NOW B(8) PORTION (B(7) MAKES NO CONTRIBUTION TO PHI-DOT.)
* * * * *
S=-LAM**2*B(8)
CALL ADDS(12,12,S)
CALL DIVV(12,3,13)
* * * * *
* * * * * THE PHI-DOT VECTOR IS NOW FORMED AND CONTAINED IN X13.
* * * * * NOW FORM THE NEW NORMALIZED X11.

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BRK15560
BRK15570
BRK15580
BRK15590
BRK15600
BRK15610
BRK15620
BRK15630
BRK15640
BRK15650
BRK15660
BRK15670
BRK15680
BRK15690
BRK15700
BRK15710
BRK15720
BRK15730
BRK15740
BRK15750
BRK15760
BRK15770
BRK15780
BRK15790
BRK15800
BRK15810
BRK15820
BRK15830
BRK15840
BRK15850
BRK15860
BRK15870
BRK15880
BRK15890
BRK15900
BRK15910
BRK15920
BRK15930
BRK15940
BRK15950
BRK15960
BRK15970
BRK15980
BRK15990
BRK16000
BRK16010
BRK16020
BRK16030

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C	SLBRoutine CIVV(N1,N2,N3)	BRK16520
	REAL*8 X(30,101),ALFA(12),G(4),S	BRK16530
	COMMON X,ALFA,G,N,KP	BRK16540
	DO 1 I=1,N	BRK16550
	1 X(N3,I)=X(N1,I)/X(N2,I)	BRK16560
	RETURN	BRK16570
	END	BRK16580
C	SUBROUTINE ADDS(N1,N2,S)	BRK16590
	REAL*8 X(30,101),ALFA(12),G(4),S	BRK16600
	COMMON X,ALFA,G,N,KP	BRK16610
	DO 1 I=1,N	BRK16620
	1 X(N2,I)=X(N1,I)+S	BRK16630
	RETURN	BRK16640
	END	BRK16650
C	SUBROUTINE SUBS(N1,N2,S)	BRK16660
	REAL*8 X(30,101),ALFA(12),G(4),S	BRK16670
	COMMON X,ALFA,G,N,KP	BRK16680
	DO 1 I=1,N	BRK16690
	1 X(N2,I)=X(N1,I)-S	BRK16700
	RETURN	BRK16710
	END	BRK16720
C	SUBROUTINE MULS(N1,N2,S)	BRK16730
	REAL*8 X(30,101),ALFA(12),G(4),S	BRK16740
	COMMON X,ALFA,G,N,KP	BRK16750
	DO 1 I=1,N	BRK16760
	1 X(N2,I)=X(N1,I)*S	BRK16770
	RETURN	BRK16780
	END	BRK16790
C	SUBROUTINE DIVS(N1,N2,S)	BRK16800
	REAL*8 X(30,101),ALFA(12),G(4),S	BRK16810
	COMMON X,ALFA,G,N,KP	BRK16820
	DO 1 I=1,N	BRK16830
	1 X(N2,I)=X(N1,I)/S	BRK16840
	RETURN	BRK16850
	END	BRK16860
C	SUBROUTINE PRIV(N1,I,J)	BRK16870
	REAL*8 X(30,101),ALFA(12),G(4),S	BRK16880
	COMMON X,ALFA,G,N,KP	BRK16890
	DO 1 I=1,N	BRK16900
	1 IF SECOND ARGUMENT EQUALS 0, GO DIRECTLY TO RETURN	BRK16910
	IF SECOND ARGUMENT EQUALS 1, PRINT THE VECTOR	BRK16920
	IF SECOND ARGUMENT EQUALS 2, PRINT THE IDENTITY AND THE VECTOR	BRK16930
	IF SECOND ARGUMENT EQUALS 3, PRINT THE VECTOR NUMBER AND THE VECTOR	BRK16940
		BRK16950
		BRK16960
		BRK16970
		BRK16980
		BRK16990

```

C IF SECOND ARGUMENT EQUALS 4; PRINT NUMBER, IDENTITY, AND VECTOR.
C IF SECOND ARGUMENT EQUALS 5; PRINT IDENTITY ONLY.
      IF(I.EQ.0) GO TO 10
      IF(I.EQ.1) GO TO 1
      IF(I.EQ.2) GO TO 2
      IF(I.EQ.3) GO TO 3
      IF(I.EQ.4) GO TO 4
      IF(I.EQ.5) GO TO 5
      DO 8 K=1,N
      WRITE(6,9) K,X(N1,K)
      9 FORMAT(30X,15,1PE20.5)
      10 RETURN
      21 WRITE(6,21) J
      21 FORMAT(//,30X,'VECTOR WITH IDENTITY ',15,' FOLLOWS:')
      3 WRITE(6,31) N1
      31 FORMAT(//,30X,'VECTOR NUMBER ',15,' FOLLOWS:')
      4 GO TO 1
      41 WRITE(6,41) N1,J
      41 FORMAT(//,30X,'VECTOR X',12,' WITH IDENTITY ',15,' FOLLOWS:')
      5 GO TO 1
      51 WRITE(6,51) J
      51 FORMAT(//,30X,'VECTOR WITH IDENTITY ',15,' HAS BEEN GENERATED.')
      GO TO 10
END
SUBROUTINE INTV(N1,N2)
REAL*8 X(30,101),ALFA(12),G(4),S,ADD,NINO,NTNO,FIVO,THTO,EN,R
COMMON X,ALFA,G,N,KP
EN=N-1
EN=1.D+0/EN
NINO=EN*9.D+0/2.4D+1
NTNO=EN*1.9D+1/2.4D+1
FIVO=EN*5.D+0/2.4D+1
THTO=EN*1.3D+1/2.4D+1
R=EN/2.4D+1
X(N2,1)=0.D+0
X(N2,2)=NINO*X(N1,1)+NTNO*X(N1,2)-FIVO*X(N1,3)+X(N1,4)*R
NM3=N-3
DO 1 K=1,NM3
  KP1=K+1
  KP2=K+2
  KP3=K+3
  ADD=THTO*(X(N1,KP1)+X(N1,KP2))-R*(X(N1,K)+X(N1,KP3))
  1 X(N2,KP2)=X(N2,KP1)+ADD
  X(N2,N)=X(N2,N-1)+NINO*X(N1,N)+NTNO*X(N1,N-1)-FIVO*X(N1,N-2)
  1+X(N1,N-3)*R
  1 RETURN
END

```

```

BRK17000
BRK17010
BRK17020
BRK17030
BRK17040
BRK17050
BRK17060
BRK17070
BRK17080
BRK17090
BRK17100
BRK17110
BRK17120
BRK17130
BRK17140
BRK17150
BRK17160
BRK17170
BRK17180
BRK17190
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BRK17220
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BRK17240
BRK17250
BRK17260
BRK17270
BRK17280
BRK17290
BRK17300
BRK17310
BRK17320
BRK17330
BRK17340
BRK17350
BRK17360
BRK17370
BRK17380
BRK17390
BRK17400
BRK17410
BRK17420
BRK17430
BRK17440
BRK17450
BRK17460
BRK17470

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C
SUBROUTINE CUPV(N1,N2)
  REAL*8 X(30,101),ALFA(12),G(4),S
  COMMON X,ALFA,G,N,KP
  DO 1 I=1,N
    1 X(N2,I)=X(N1,I)
  RETURN
END
SUBROUTINE NORV(N1)
  REAL*8 X(30,101),ALFA(12),G(4),S
  COMMON X,ALFA,G,N,KP
  CALL INTV(N1,20)
  IF(DABS(X(20,N)) .GT. 1.D-10) GO TO 2
  DO 1 I=1,N
    1 X(19,I)=DABS(X(N1,I))
  CALL INTV(19,20)
  2 CALL DIVS(N1,18,X(20,N))
  CALL DUPV(18,N1)
  RETURN
END
SUBROUTINE QVEC(I,J)
  REAL*8 X(30,101),ALFA(12),G(4),S
  COMMON X,ALFA,G,N,KP
  CALL MJLV(5,1,20)
  CALL INTV(20,19)
  CALL MJLV(8,1,20)
  CALL ADDV(19,20,J)
  RETURN
END
C

```

```

BRK17480
BRK17490
BRK17500
BRK17510
BRK17520
BRK17530
BRK17540
BRK17550
BRK17560
BRK17570
BRK17580
BRK17590
BRK17600
BRK17610
BRK17620
BRK17630
BRK17640
BRK17650
BRK17660
BRK17670
BRK17680
BRK17690
BRK17700
BRK17710
BRK17720
BRK17730
BRK17740
BRK17750
BRK17760
BRK17770
BRK17780
BRK17790

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